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A Class of Solutions for Multiperson Multicriteria Decision Making

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Summary. A model for multiperson multicriteria decision making is proposed, in which each individual is characterized by a weight together with a vector expressing that individual's position or preferences with respect to a number of criteria. Next, a one-parameter class of solutions is developed, of which specific members bear a strong resemblance to concepts from statistics. Our approach is related to the potential function approach in physics.

Zusammenfassung. Ein Modell für Mehrpersonen-Mehrkriterien-Entscheidungsprobleme wird vorgeschlagen, indem jedes Individuum charakterisiert wird von einem Gewicht und gleichzeitig von einem Kriterien-Vektor. Eine Familie von Lösungen, parameterisiert von einem Parameter, wird entwickelt, wobei bestimmte Lösungen in dieser Familie bestimmten statistischen Konzepten entprechen. Unser Ansatz weist Parallelen mit dem Ansatz der Potentialfunktion in der Physik auf.

1. Introduction

Consider a situation where a number of owners (directors, shareholders) of a firm have to decide on how to divide an amount of money over a number of investment projects. Or a situation in which voters have to choose among a number of candidates. Also, imagine a situation where some politicians or political parties have to reach a collective decision on, say, a road to be built or not, and where each politician has his or her own private judgement based on the weights he or she attaches to a number of criteria. All these – and many more – situations are examples of multiperson multicriteria decision making, characterized by a similar formal structure.

Such a formal structure, capturing many possible applications, is described and studied in this paper. More precisely, there will be n individuals, and each individual will be characterized by some positive weight as well as a vector of length m describing that individual's position, or preferences, with respect to m criteria. We will use the term solution for a map assigning to such an n-person m-criteria decision problem a set of m-vectors (preferably consisting of one vector), to be interpreted as compromise vectors.

Thus, we have a model involving multiple parties with conflicting interests. We make the informal assumption that there is some institution that proposes a solution, which is then binding: this institution, however, may be taken in a broad sense, and it may consist of the individuals themselves. We further assume, in this paper, that there is no coalition formation. Consequently, our model is closely related to the models studied in social choice theory and in (axiomatic, cooperative) bargaining game theory. In these areas, the standard practice is to determine solutions on the ground of two main criteria. Firstly, it should be possible to provide a reasonable axiomatic foundation for a solution. Secondly, preferably the solution should be intuitive and attractive by itself, i.e., as a formula. Historically, in game theory as well as in social choice theory, many solutions were originally proposed as intuitively attractive concepts, and axiomatically characterized afterwards.

In the present paper, we do not take an axiomatic approach, but rather pay attention to this second criterion. Specifically, we will formulate a class of solutions parametrized by one parameter, allowing us to vary the extent to which individual power is to be taken into account. Certain choices of the parameter will lead to solutions related to statistical concepts such as median, mean, and mode, and to the min max regret concept from decision theory. We will further show how our approach is inspired by and related to the use of the potential function in physics.

The remainder of the paper is organized as follows. In Sect. 2, we describe the model and, formally, the various applications hinted at in the beginning of this introduction. Section 3 describes our one-parameter family of solutions, and the main results concerning these solutions. All the proofs are collected in Sect. 4, and Sect. 5 concludes.

The reader will have noticed that, up till now, we did not include any references to the literature. To the best of our knowledge, what we do is new, but of course not unrelated to existing literature. Therefore, we restrict ourselves at this point to a selection of a few general works on the mentioned areas: Fandel (1972) on multicriteria analysis, Hwang and Lin (1987) on multicriteria analysis, French (1986) on decision making. We apologize in advance to all authors who should also have been mentioned.

2. The Model and Applications

An *n-person m-criteria decision problem* is an ordered $2n$-tuple $\langle w^i, \varrho^i \rangle_{i=1,2,\dots,n}$ where, for each $i \in N := \{1,2,\dots,n\}$, w^i is an element of $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_j \geq 0, j=1,2,\dots,m\}$, and ϱ^i is a positive real number. The general interpretation of such a problem is as follows. Each individual $i \in N$ is characterized by the vector w^i expressing the weights that individual attaches to m criteria, and the number ϱ^i interpreted as an indication of that individual's power or influence.

An investment problem. Each individual is a shareholder of the same firm. The number ϱ^i expresses that individual's power, for which the percentage of shares is a natural candidate. The vector w^i states how much money individual i would invest in each of m projects, provided he/she were to decide by himself/herself. The problem is to find a compromise vector expressing to a smaller or larger extent the individual powers and preferences. In other words, one wants a final answer to the question how much money to invest in each project.

Another (open-ended) interpretation could go as follows. Each individual i contributes an amount of money ϱ^i. The total sum $\Sigma_{i=1}^n \varrho^i$, is then to be invested in m projects. Thereby, the vector w^i may be interpreted in several ways. One way would be to think of w^i_j as giving the return or utility to individual i of an investment of 1 unit of money in project j, assuming that these utilities are linear and additive, and possibly relative to a 0-criterion (project) with fixed weight (utility) 1 suppressed from notation.

A political decision problem. Consider n politicians who have to reach a unanimous decision on whether to build a road or not. Ultimately, the decision will be taken on the basis of the weights assigned to a number of criteria, such as economical importance and environmental damage. The modeling of this decision is a problem in itself, preceded by the problem of finding a compromise vector of weights attached to the different criteria, which belongs to the realm of this paper. The numbers ϱ^i might reflect the strengths or sizes of the parties of the politicians. Again, the weights of the criteria may be interpreted as relative to the fixed weight, e.g. 1, of some 0-criterion.

A voting problem. There are n voters. Voter i has ϱ^i votes (absolute, or as a percentage). The vector w^i describes the way in which individual i ranks the m candidates according to strength of preference, so the numbers w^i_j may be taken relative again. Again, one looks for a compromise vector ranking the m candidates (eventually, the candidate with the highest number might be elected, or some other procedure followed). A natural way to reach such a compromise vector may be to add all the votes per candidate after first distributing the individual votes according to the individual rankings.

A prediction problem. The m criteria are now m states of the economy, exactly one of which will be true. There are n wise economists, ϱ^i being a measure of economist i's wisdom. Economist i attaches probability w^i_j to state of the economy j being the true state. How to find a compromise prediction?

These examples indicate that, with some flexibility of interpretation, a variety of situations may be described by the same formal structure. They also show that one of the main problems in mathematical modeling (of this kind, but in general as well) is to choose the "right" scaling or normalizations. We take the ϱ^i to be any positive numbers, and the w^i to be any nonnegative vectors, since we have no reason to choose any particular (other) normalization. Of course, one might want to choose a specific normalization depending on the specific context at hand, e.g. the coordinates of the w^i summing to 1 in the prediction problem above. The results in this paper will be valid also under such a normalization.

We conclude this section with the definition of a solution. Let $\mathscr{P}$ denote the collection of all n-person m-criteria decision problems. Unless stated otherwise, P is used to denote the typical problem $\langle w^i, \varrho^i \rangle_{i=1,2,\dots,n} \in \mathscr{P}$. A *solution* is a correspondence $\varphi : \mathscr{P} \to \mathbb{R}^m_+$ assigning to each problem $P \in \mathscr{P}$ a nonempty set of "compromise" vectors $\varphi(P)$. In the next section, we will mainly consider a specific one-parameter family of solutions. In many interesting cases, these solutions will be single-valued.

3. Solutions to the Multiperson Multicriteria Problem

We will first describe four solutions for the n-person m-criteria decision problem. Later on we will show that these solutions are particular instances from a one-parameter family of solutions. This will already be reflected in the notations we use for these solutions.

The *mode solution* φ^0 is defined $\varphi^0(P) = \{w^i : \Sigma_{j: w^j = w^i} \varrho^j \geq \Sigma_{j: w^j = w^k} \varrho^j$ for every $k \neq i\}$. This solution takes only the power indices ϱ^i into consideration, not the weight vectors w^i. It obeys the most powerful individual(s), where it is understood that power indices are added if weight vectors coincide. It takes its name from the statistical concept of mode: the analogy becomes transparent if we interpret ϱ^i as the frequency of observing the "value" w^i of some stochastic variable in a random experiment. In some contexts it may be a very reasonable solution. For instance, if we take the voting example with each voter giving all votes to the most preferred candidate, then the mode solution simply reflects majority voting.

The *median solution* φ^1 is defined by $\varphi^1(P) = \{z \in \mathbb{R}^m_+ : z$ minimizes $\Sigma_{i=1}^n \varrho^i \|x - w^i\|_2\}$, where $\|\cdot\|_2$ denotes the Euclidean norm: $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_m^2}$. That this solution is well-defined is not hard to see, but of later concern.

It is related to the statistical concept of median: also this will be argued later. Compromises assigned by φ^1 minimize the power-weighted sum of Euclidean distances to the weight vectors.

The *mean solution* φ^2 is defined by $\varphi^2(P)=\{\Sigma_{i=1}^n \varrho^i w^i / \Sigma_{i=1}^n \varrho^i\}$. This solution assigns to a problem the power weighted average of the weight vectors. It is clearly analogous to the statistical concept of mean if we interpret again ϱ^i as the frequency of observing w^i in a random experiment.

The *minmax solution* φ^∞ is defined by $\varphi^\infty(P)=\{z \in \mathbb{R}_+^m : \max_{i \in N} \| z-w^i \|_2 = \min_{x \in \mathbb{R}_+^m} \max_{i \in N} \| x-w^i \|_2\}$. Also this solution is single-valued, as will be shown later. It does not take the power indices ϱ^i into account but just minimizes maximal distance to the individual weight vectors. In spirit, it is akin to Rawls's maximin principle, see Rawls (1971). It is analogous to the minimax regret concept in (one-person) decision making under uncertainty, see e.g. French (1986, p. 37), or Savage (1951).

The main insight of this paper is that these four solutions can be obtained as particular instances from a whole family of solutions. But first we start, more generally, by considering functions $U_f: \mathbb{R}^m \to \mathbb{R}$ of the form

$$U_f(x)=\sum_{i=1}^n \varrho^i f(\| x-w^i \|) \tag{1}$$

for a given problem $P \in \mathscr{P}$, where $f:[0,\infty)\to\mathbb{R}$ is a continuous monotonically strictly increasing function, and $\|\cdot\|$ is a norm on $\mathbb{R}^m$. The value $U_f(x)$ is interpreted as measuring the potential conflict between, or aggregate disutility of, the individuals resulting from imposing a compromise vector $x \in \mathbb{R}^m$. It is our (or the individuals', or an arbitrating institution's) objective to find a compromise which minimizes this function. Readers familiar with physics will recognize the close analogy between this approach and the potential function approach in physics. We will digress on this analogy later on this section. Here, we proceed by collecting some facts on the function U_f in the following theorem. All proofs are delayed until Sect. 4.

Theorem 1. *Let U_f be the function defined in (1). Then:*
(a) U_f has a global minimum.
(b) If f is convex, then U_f is convex, and hence the global minimum locations of U_f form a convex set.
(c) If f is strictly convex the U_f is also strictly convex and hence U_f has a unique global minimum location.
If, furthermore, $\|\cdot\|$ is the Euclidean norm, then:
(d) Every global minimum location of U_f is an element of the convex hull of the points w^i $(i \in N)$.
(e) If f is convex and U_f has two different global minimum locations x and y, then every point w^i is an element of the straight line through x and y.

We will be particularly interested in functions f of the form $f(y)=y^\alpha$, $\alpha>0$. For convenience, we use the notation U^α instead of U_f if $f(y)=y^\alpha$ and $\|\cdot\|$ is the Euclidean norm. So

$$U^\alpha(x)=\sum_{i=1}^n \varrho^i \| x-w^i \|_2^\alpha, \qquad x \in \mathbb{R}^m.$$

Note that $f(y)=y^\alpha$ is convex if and only if $\alpha \geq 1$, and strictly convex if and only if $\alpha>1$. So, in view of Theorem 1, U^α has a unique minimum location situated in the convex hull of $\{w^i : i \in N\}$ whenever $\alpha>1$. If $\alpha=1$, the global minimum location does not have to be unique; however, if the points w^i are not collinear, then also in that case the global minimum location will be unique. If $0<\alpha<1$, then still U^α has a global minimum, but it does not have to be attained at a unique point. We further define $U^0(x)=\lim_{\alpha\to 0} U^\alpha(x)$ for $x \in \mathbb{R}^m$. The following sequence of theorems shows that the solutions for the n-person m-criteria decision problem defined above, namely the mode, median, mean, and minmax solutions φ^0, φ^1, φ^2 and φ^∞, respectively, can be obtained by considering the sets of global minimum locations of U^0, U^1, U^2, and the limit of minimum locations of U^α for $\alpha\to\infty$, respectively.

Theorem 2. *$\varphi^0(P)$ is the set of global minimum locations of the function U^0.*

It follows from the (simple) proof of Theorem 2 that it would hold for any other norm instead of the Euclidean norm.

Theorem 3. *(a) $\varphi^1(P)$ is the set of global minimum locations of U^1.*
(b) Let v^α denote the (unique) global minimum location of U^α for $\alpha>1$. Then $v^1 := \lim_{\alpha\downarrow 1} v^\alpha$ exists and $v^1 \in \varphi^1(P)$.
(c) If not all w^i are on the same straight line, then $\varphi^1(P)=\{v^1\}$.
(d) Suppose all w^i are on the same line, say $w^i=p+\lambda_i(q-p)$ for points $p \neq q$ on that line and $\lambda_i \in \mathbb{R}$. Then: either there exist $k \in N$ and $0<\lambda<1$ such that $\Sigma_{i:\lambda_i<\lambda_k}\varrho^i+\lambda\Sigma_{i:\lambda_i=\lambda_k}\varrho^i=\Sigma_{i:\lambda^i>\lambda^k}\varrho^i+(1-\lambda)\Sigma_{i:\lambda^i=\lambda^k}\varrho^i$, or there exist adjacent w^k and w^l $(\lambda_k<\lambda_l)$ with $\Sigma_{i:\lambda_i\leq\lambda_k}\varrho^i=\Sigma_{i:\lambda_i\geq\lambda_l}\varrho^i$.

In the former case, $\varphi^1(P)=\{w^k\}$, while in the latter case $\varphi^1(P)$ is the line segment connecting w^k and w^l. In the latter case, the point v^1 is determined by the formula $\Pi_{i:\lambda_i\leq\lambda_k}\|v^1-w^i\|_2^{\varrho^i}=\Pi_{i:\lambda_i\geq\lambda_l}\|v^1-w^i\|_2^{\varrho^i}$.

Note that (as we have noted before) in "most" cases φ^1 assigns a unique point to a problem P: see part *(c)* of Theorem 3. Part *(d)* describes the situation in which all the criteria vectors w^i are on the same straight line. It is because of this situation that we have assigned the name "median solution" to φ^1: again, interpret ϱ^i as the frequency of observing w^i, where the points w^i are ordered along a straight line. Note that *(d)* offers a way to select a unique point from the median in the case of nonuniqueness, namely the point v^1 (for which a formula is given as well). Further, in the general case where the w^i are not on the same line, the unique element of $\varphi^1(P)$ can be viewed as a generalization of the median: as such, the concept might be useful in multivariate statistics where the outcomes of a random variable are not ordered along a straight line.

Theorem 4. *$\varphi^2(P)$ consists of the unique global minimum location of U^2.*

Theorem 5. *$v^\infty := \lim_{\alpha\to\infty} v^\alpha$ exists, and $\{v^\infty\} = \varphi^\infty(P)$.*

Theorem 5 states that the global minimum locations of the functions U^α for α going to infinity converge to a unique point, namely the point assigned by the minmax solution φ^∞. Indeed, at this point the power indices of the individuals do not matter any more, only the distances to the criteria vectors are relevant. The other extreme was the case $\alpha=0$ where only the power indices of the individuals were important. So any $0\le\alpha\le\infty$ is a compromise between individual power and individual regret, the former being maximal for $\alpha=0$, the maximum of the latter minimal for $\alpha\to\infty$. It should be noted that Theorem 5 holds for a larger class of norms: this is proved in Sect. 4.

Let us summarize our main findings. We have introduced a family of functions U^α, $0\le\alpha<\infty$, which can be interpreted as measuring the degree of conflict, or aggregate disutility, between the individuals in a multiperson multicriteria decision problem. For any α, the set of global minimum locations of U^α determines a solution to the problem. For $\alpha=0,1,2$, and approaching infinity, we obtain the intuitively appealing solutions called mode, median, mean, and minmax, respectively, in view of the analogy with descriptive statistics. Particular choices of α will depend on the context of application. We already mentioned a few times that our approach is closely related to the potential function approach in physics; recently, a similar inspiration has been fruitful in game theory, see Mas-Colell and Hart (1989). An earlier application of the potential function approach to game theory can be found in Spinetto (1971). We conclude this section with a digression on the potential function in physics, and a few physical analogies of our context.

Digression on the Use of the Potential in Physics

(a) Gravitation. Let w^i $(i\in N)$ be points with mass ϱ^i. The field of gravitation is described by the potential $G(x)=-\Sigma_{i\in N}\varrho^i\|x-w^i\|_2^{-1}$, for $x\in\mathbb{R}^3$. For $x\notin\{w^i: i\in N\}$, the gradient vector of G is $\nabla G(x)=-\Sigma_{i\in N}\varrho^i\dfrac{w^i-x}{\|x-w^i\|_2^3}$ and gives the forces exerted on x by the point masses w^i. If $\nabla G(x)=0$, then x is a stationary point; in any direction, the potential is constant, and consequently a point mass at x will not move.
(b) An electric field. The potential $E(x)=\Sigma_{i\in N}\varrho^i\|x-w^i\|_2^{-1}$ with gradient $\nabla E(x)=\Sigma_{i\in N}\varrho^i\dfrac{w^i-x}{\|x-w^i\|_2^3}$ describes the situation where the w^i are particles with positive charge ϱ^i. Again, the gradient vector describes the electric forces exerted on a positively charged particle at x.

The following two examples present physical analogies of the median solution φ^1 and the mean solution φ^2.

(c) Holes in a plane. Let w^i be the coordinates of a hole in a horizontal plane in a standard gravitational field (e.g. somewhere on the earth).

A point mass is situated on this plane at point x. Thread i connects this point mass to another point mass sized ϱ^i, via the hole at $w^i (i\in N)$; the latter is hanging below w^i, and therefore exerts a force on x with size ϱ^i, in the direction of w^i.

If x doesn't coincide with one of the w^i, the total force exerted on x equals $\Sigma_{i\in N}\varrho^i\dfrac{w^i-x}{\|w^i-x\|_2}$. This force corresponds to the potential $G(x)=\Sigma_{i\in N}\varrho^i\|w^i-x\|_2$ (which function is defined at the w^i too). We conclude that this physical model corresponds to the median solution φ^1.
(d) Helical springs. Let ϱ_i be the spring constant of helical spring i $(i\in N)$, which is connected to fixed point w^i at one side, and to a point mass at point x at the other side (so all springs are connected to each other at x). The force exerted on point x now equals: $\Sigma_{i\in N}\varrho^i(w^i-x)$. The corresponding potential is equal to: $G(x)=\Sigma_{i\in N}\varrho^i\|x-w^i\|_2^2$. If $\nabla G(x)=0$, which is the case when $x=\dfrac{\Sigma_{i\in N}\varrho^i w^i}{\Sigma_{i\in N}\varrho^i}$, x is a stationary point.

So we conclude that this physical model corresponds to the mean solution φ^2.

4. Proofs and Auxiliary Results

This section contains no new material. It presents the proofs of the results in the previous section, together with some auxiliary results and a few extensions; a reader not interested in these proofs may skip the section.

The following elementary lemma will be needed in the sequel. We state it without proof. Recall that $\|\cdot\|_2$ denotes the Euclidean norm.

Lemma 1. *Let $a, b\in\mathbb{R}^n$ with $\|a+b\|_2=\|a\|_2+\|b\|_2$. Then $a=0$ or $b=\mu a$ for some $\mu\in[0,\infty)$.*

Proof of Theorem 1. (a) Choose $\Delta>0$ such that $\|w^i\|\le\Delta$ for all $i\in N$. Let $x\in\mathbb{R}^m$ with $\|x\|>2\Delta$. Then, for every $i\in N$: $\|x-w^i\|\ge\|x\|-\|w^i\|>\Delta$, hence $U_f(x)=\Sigma_{i=1}^n\varrho^i f(\|x-w^i\|)>\Sigma_{i=1}^n\varrho^i f(\Delta)\ge\Sigma_{i=1}^n\varrho^i f(\|0-w^i\|)=U_f(0)$, where we use the fact that f is monotonically strictly increasing. So $\inf_{x\in\mathbb{R}^m}U_f(x)=\inf_{x\in B_{2\Delta}(0)}U_f(x)$ where $B_{2\Delta}(0)=\{x\in\mathbb{R}^m: \|x\|\le 2\Delta\}$. Since $B_{2\Delta}(0)$ is compact and U_f continuous as a composition of two continuous functions, we know that U_f has a global minimum on $B_{2\Delta}(0)$ and hence on $\mathbb{R}^m$.
(b) The second statement follows immediately from the first statement. For the first statement, let x, $y\in\mathbb{R}^m$ and $0<\lambda<1$. Then $U_f(\lambda x+(1-\lambda)y)=\Sigma_{i=1}^n\varrho^i f(\|\lambda x+(1-\lambda)y-w^i\|)=\Sigma_{i=1}^n\varrho^i f(\|\lambda(x-w^i)+(1-\lambda)(y-w^i)\|)\le\Sigma_{i=1}^n\varrho^i f(\|\lambda(x-w^i)\|+\|(1-\lambda)(y-w^i)\|)\le\Sigma_{i=1}^n\varrho^i(\lambda f(\|x-w^i\|)+(1-\lambda)f(\|y-w^i\|))=\lambda U_f(x)+(1-\lambda)U_f(y)$.
(c) If f is strictly convex, then the second inequality in the proof in *(b)* is strict, hence U_f is strictly convex.

(d) Suppose $v \in \mathbb{R}^m$ is not an element of the convex hull W of $\{w^i : i \in N\}$. Then, by a well-known separation theorem from convex analysis, there exists $p \in \mathbb{R}^m$, $p \neq 0$, with $p \cdot v < \min_{x \in W} p \cdot x$ (the dot denotes the usual inner product on $\mathbb{R}^m$). Let $c := \min_{x \in W} p \cdot x$, and $d := v + \frac{c - p \cdot v}{p \cdot p} \cdot p$. Then $p \cdot d = c \leq p \cdot x$ for all $x \in W$, in particular $p \cdot d \leq p \cdot w^i$ for all $i \in N$. Now $\|d - w^i\|_2^2 = \|v - w^i + \frac{c - p \cdot v}{p \cdot p} \cdot p\|_2^2 = \|v + w^i\|_2^2 + \frac{(c - p \cdot v)^2}{p \cdot p} + 2 \frac{c - p \cdot v}{p \cdot p} p \cdot (v - w^i) = \|v - w^i\|_2^2 + \frac{c - p \cdot v}{p \cdot p} \ (c - p \cdot v + 2p \cdot (v - w^i)) = \|v - w^i\|_2^2 + \frac{c - p \cdot v}{p \cdot p} ((c - p \cdot w^i) + (p \cdot v - p \cdot w^i)) < \|v - w^i\|_2^2$. So, for every $i \in N$, $\|d - w^i\|_2 < \|v - w^i\|_2$. Now $U_f(d) = \Sigma_{i=1}^n \varrho^i f(\|d - w^i\|_2) < \Sigma_{i=1}^n \varrho^i f(\|v - w^i\|_2) = U_f(v)$, so v cannot be a global minimum location of U_f. Hence any global minimum location must be in the convex hull of $\{w^i : i \in N\}$.

(e) Let $0 < \lambda < 1$, then $U_f(\lambda x + (1-\lambda)y) = U_f(x) = U_f(y)$, by part *(b)* of this theorem. Hence, in the proof of part *(b)* we must have only equalities in this case. In particular, it follows that for all $i \in N$ we have $\|\lambda(x - w^i) + (1-\lambda)(y - w^i)\|_2 = \|\lambda(x - w^i)\|_2 + \|(1-\lambda)(y - w^i)\|_2$. Applying lemma 1 and rearranging terms, we find that for some number $\mu \geq 0$ we have: $w^i = \alpha x + \beta y$, where $\alpha = \frac{\lambda}{\lambda - \mu + \lambda\mu}$ and $\beta = \frac{-\mu + \lambda\mu}{\lambda - \mu + \lambda\mu}$. Since $\alpha + \beta = 1$, we conclude that w^i is an element of the straight line through x and y. This holds for any $i \in N$. □

Parts *(d)* and *(e)* of Theorem 1 indeed do not hold for arbitrary norms. For instance, let $N = \{1, 2\}$, $m = 2$, $w^1 = (0, 0)$, $w^2 = (1, 0)$, $\varrho^1 = \varrho^2 = 1$, $\|x\| = \max\{|x_1|, |x_2|\}$ for all $x \in \mathbb{R}^2$, and let f be identity. Then the set of global minimum locations of the corresponding function U_f is the convex hull of $\left\{w^1, w^2, \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)\right\}$. So neither *(d)* nor *(e)* hold for this norm.

Proof of Theorem 2. Obviously, $U^0(x) = \lim_{\alpha \to 0} U^\alpha(x) = \Sigma_{i: x \neq w^i} \varrho^i$. So $\varphi^0(P) = \{w^i : \Sigma_{w^j = w^i} \varrho^j \geq \Sigma_{w^j = w^k} \varrho^j$ for every $k \neq i\}$ is the set of global minimum locations of U^0. □

Before proving Theorem 3, we need the following lemma. Some additional notation: let $g : [0, \infty) \to \mathbb{R}_+^m$ be a function assigning to every $\alpha \geq 0$ a global minimum location of U^α.

Lemma 2. *$U^\alpha(g(a))$ is a continuous function of α on $(0, \infty)$.*

Remark. It can be shown that $U^\alpha(g(\alpha))$ is also continuous at $\alpha = 0$; this however, needs an additional proof (since U^0 is not continuous), which we omit since we do not need the result in the sequel.

Proof of Lemma 2. Suppose $U^\alpha(g(\alpha))$ is not continuous at some $\beta > 0$. We will show that this leads to a contradiction. Define $\tilde{U}(x, \alpha) = U^\alpha(x)$ for $\alpha \in (0, \infty)$ and $x \in \mathbb{R}^m$. Then $\tilde{U}$ is continuous on $\mathbb{R}^m \times (0, \infty)$. Since, by assumption, $\tilde{U}(g(\alpha), \alpha)$ is not continuous at β, there is a sequence $\alpha_1, \alpha_2, \ldots$ with $\lim_{k \to \infty} \alpha_k = \beta$ and an $\varepsilon > 0$ such that $|\tilde{U}(g(\alpha_k), \alpha_k) - \tilde{U}(g(\beta), \beta)| > \varepsilon$ for all $k \in \mathbb{N}$. Since, by Theorem 1 (d), for all $k \in \mathbb{N}$, $g(\alpha_k)$ is an element of the convex hull of $\{w^i : i \in N\}$, which is a compact set, there is a converging subsequence, say $g(\beta_k)$ and $z := \lim_{k \to \infty} g(\beta_k)$. Then $\lim_{k \to \infty} (g(\beta_k), \beta_k) = (z, \beta)$ and $\tilde{U}(z, \beta) = \lim_{k \to \infty} \tilde{U}(g(\beta_k), \beta_k) \leq \lim_{k \to \infty} \tilde{U}(g(\beta), \beta_k) = \tilde{U}(g(\beta), \beta)$. By definition of $g(\beta)$ we also have $\tilde{U}(z, b) \geq \tilde{U}(g(\beta), \beta)$, hence $\tilde{U}(z, b) = \tilde{U}(g(\beta), \beta)$ and so $\lim_{k \to \infty} \tilde{U}(g(\beta_k), \beta_k) = \tilde{U}(g(\beta), \beta)$, a contradiction since $\{\beta_k\}$ is a subsequence of $\{\alpha_k\}$. □

Proof of Theorem 3. *(a)* follows by definition.

We now first prove the statements in *(d)*, except for the last statement there. Let p, q, and $\lambda^i (i \in N)$ as in *(d)*. Let further x and y be global minimum locations of U^1 and suppose $w^j = \lambda x + (1-\lambda)y$ for some $j \in N$ and $0 < \lambda < 1$. Apply the argument used in the proof of Theorem 1 *(b)* to these points x and y: since x and y are global minimum locations, it follows that we must have equality signs there everywhere, and in particular $\|\lambda(x - w^j) + (1-\lambda)(y - w^j)\|_2 = \|\lambda(x - w^j)\|_2 + \|(1-\lambda)(y - w^j)\|_2$. Since the lefthandside expression must be 0, we have $x = w^j = y$. We conclude that the (convex) set of global minimum locations of U^1 must be a subset of a line segment with two adjacent elements of $\{w^i : i \in N\}$ as endpoints, say w^k and w^l with $\lambda_k < \lambda_l$. For the global minimum location x in the convex hull of $\{w^k, w^l\}$, it is easy to show that: $U^1(x) = \Sigma_{i \in N} \varrho^i \|x - w^i\|_2 = \Sigma_{i: \lambda_i \leq \lambda_k} \varrho^i \|x - w^k\|_2 + \Sigma_{i: \lambda_i \geq \lambda_l} \varrho^i \|x - w^l\|_2$. If $\Sigma_{i: \lambda_i \leq \lambda_k} \varrho^i = \Sigma_{i: \lambda_i \geq \lambda_l} \varrho^i$, then U^1 is constant on the convex hull of $\{w^k, w^l\}$ which implies that this is exactly the set of global minimum locations. Otherwise, we have, say, $\Sigma_{i: \lambda_i \leq \lambda_k} \varrho^i > \Sigma_{i: \lambda_i \geq \lambda_l} \varrho^i$ and hence $x = w^k$ must be the unique global minimum location. Since, by a similar argument, $\Sigma_{i: \lambda_i < \lambda_k} \varrho^i < \Sigma_{i: \lambda_i \geq \lambda_k} \varrho^i$, there must exist a λ as in statement *(d)*. We have proven *(d)* except for the last statement.

We continue by proving the final statement in *(d)*. Let V denote the convex hull of w^k and w^l, that is, the set of global minimum locations of U^1. For every $i \in N$ let μ_i be defined by $w^i = w^k + \mu_i(w^l - w^k)$. We denote $I^+ = \{i \in N : \mu_i \geq 1\}$ and $I^- = \{i \in N : \mu_i \leq 0\}$. For $\mu \in [0, 1]$ and $z = w^k + \mu(w^l - w^k) \in V$ we have: $\Pi_{i \in I^-} \|z - w^i\|_2^{\varrho^i} = \|w^l - w^k\|_2^{\Sigma_{i \in I^-} \varrho^i} \ \Pi_{i \in I^-} (\mu - \mu_i)^{\varrho^i}$ which is a continuous strictly monotonically increasing function of μ on $[0, 1]$ vanishing at $\mu = 0$. Analogously, $\Pi_{i \in I_+} \|z - w^i\|_2^{\varrho^i}$ is a continuous strictly monotonically decreasing function of μ on $[0, 1]$ vanishing at $\mu = 1$. This implies that there is exactly one $\tilde{z} \in V$ with $\Pi_{i \in I^-} \|\tilde{z} - w^i\|_2^{\varrho^i} = \Pi_{i \in I^+} \|\tilde{z} - w^i\|_2^{\varrho^i}$. Moreover, $\tilde{z} \neq w^k, w^l$.

Next, let the function $H : \mathbb{R}^m \to \mathbb{R}$ be defined by $H(x) := \Sigma_{i \in J(x)} \varrho^i \ln(\|x - w^i\|_2) \|x - w^i\|_2$, where $J(x) := \{i \in N : w^i \neq x\}$. Since, in particular, $\lim_{t \downarrow 0} t \ln t = 0$, H is continuous on $\mathbb{R}^m$. Let $x = w^k + \mu(w^l - w^k)$, $0 < \mu < 1$. Then $H(x) = H(w^k + \mu(w^l - w^k z)) = \|w^l - w^k\|_2 \Sigma_{i \in I^-} \varrho^i (\mu - \mu_i)$

$\ln(\|w^l - w^k\|_2(\mu - \mu_i)) - \|w^l - w^k\|_2 \Sigma_{i\in I^+} \varrho^i(\mu - \mu_i)$ $\ln(\|w^l - w^k\|_2(\mu_i - \mu))$. So, for $\mu \in (0,1)$, $\frac{d}{d\mu} H(w^k + \mu(w^l - w^k)) = \|w^l - w^k\|_2 \Sigma_{i\in I^-} \varrho^i(\ln(\|w^l - w^k\|_2(\mu - \mu_i)) + 1) - \|w^l - w^k\|_2 \Sigma_{i\in I^+} \varrho^i(\ln(\|w^l - w^k\|_2(\mu_i - \mu)) + 1)$. Since $\Sigma_{i\in I^-} \varrho^i = \Sigma_{i\in I^+} \varrho^i$, this reduces to:

$$\frac{d}{d\mu} H(w^k + \mu(w^l - w^k))$$
$$= \|w^l - w^k\|_2 \sum_{i\in I^-} \varrho^i \ln(\|w^l - w^k\|_2(\mu - \mu_i))$$
$$= - \|w^l - w^k\|_2 \sum_{i\in I^+} \varrho^i \ln(\|w^l - w^k\|_2(\mu_i - \mu)),$$

and this function is negative for μ close to 0, positive for μ close to 1, continuous and strictly monotonically increasing on $(0,1)$. So there is exactly one point $\tilde{x} = w^k + \tilde{\mu}(w^l - w^k)$ with $0 < \tilde{\mu} < 1$ where H takes a minimum on V. From $\frac{d}{d\mu} H(\tilde{x}) = 0$, it follows that $\Sigma_{i\in I^-} \varrho^i \ln(\|w^l - w^k\|_2(\tilde{\mu} - \mu_i)) = \Sigma_{i\in I^+} \varrho^i \ln(\|w^l - w^k\|_2(\mu_i - \tilde{\mu}))$, hence $\Sigma_{i\in I^-} \varrho^i \ln \|\tilde{x} - w^i\|_2 = \Sigma_{i\in I^+} \varrho^i \ln \|\tilde{x} - w^i\|_2$. So $\tilde{x} = \tilde{z}$. In other words, $\tilde{z}$ is the unique minimum location of H on V. Further, $\frac{d}{d\alpha} U^\alpha(x) = \frac{d}{d\alpha} \Sigma_{i\in I} \varrho^i \|x - w^i\|_2^\alpha = \Sigma_{i\in J(x)} \varrho^i \ln(\|x - w^i\|_2) \|x - w^i\|_2^\alpha$, hence $H(x) = \frac{d}{d\alpha} U^\alpha(x)|_{\alpha=1}$. So $H(x) = \lim_{\varepsilon\downarrow 0} \frac{U^{1+\varepsilon} - U^1(x)}{\varepsilon}$.

The proof of *(d)* is finished if we show that $\lim_{\alpha\downarrow 1} v^\alpha = \tilde{z}$, i.e., $\lim_{\alpha\downarrow 1} g(\alpha) = \tilde{z}$. We suppose this is not the case and will derive a contradiction. If not $\lim_{\alpha\downarrow 1} g(\alpha) = \tilde{z}$, there is an $\varepsilon > 0$ and a sequence $\sigma_1, \sigma_2, \ldots$ with $\lim_{k\to\infty} \sigma_k = 0$ such that $\lim_{k\to\infty} g(1+\sigma_k) = y$ for some y on the straight line through the w^i and such that $\|g(1+\sigma_k) - \tilde{z}\|_2 > \varepsilon$ for all k (recall Theorem 1 *(d)*). In particular, $\|y - \tilde{z}\|_2 \geq \varepsilon$. By Lemma 2, we have: $U^1(g(1)) = \lim_{k\to\infty} U^{1+\sigma_k}(g(1+\sigma_k)) = \Sigma_{i\in I} \varrho^i \lim_{k\to\infty} \|g(1+\sigma_k) - w^i\|_2^{1+\sigma_k} = \Sigma_{i\in I} \varrho^i \lim_{k\to\infty} \|g(1+\sigma_k) - w^i\|_2 = U^1(y)$. So $y \in V$, and $H(y) > H(\tilde{z})$ since $\tilde{z}$ is the unique minimum location of H on V. Let $c := \frac{1}{4}(H(y) - H(c))$. Since H is continuous, there exists $\delta > 0$ such that $\|x - y\|_2 \leq \delta \Rightarrow H(x) \geq H(y) - c$ for all $x \in \mathbb{R}^m$. Let $k_0 \in \mathbb{N}$ such that $k \geq k_0 \Rightarrow \|g(1+\sigma_k) - y\|_2 < \delta$, and let $r := y + \delta \frac{\tilde{z} - y}{\|\tilde{z} - y\|_2}$. Then $\|r - y\|_2 = \delta$.

For all $k \geq k_0$, r is a convex combination of $g(1+\sigma_k)$ and $\tilde{z}$; since $U^{1+\sigma_k}(g(1+\sigma_k)) < U^{1+\sigma_k}(\tilde{z})$ and $U^{1+\sigma_k}$ is strictly convex, we have: $U^{1+\sigma_k}(r) < U^{1+\sigma_k}(\tilde{z})$ for all $k \geq k_0$. Also, $H(r) \geq H(y) - c = H(\tilde{z}) + 3c$, and $U^1(r) = U^1(\tilde{z}) = U^1(y)$ since r is a convex combination of y and $\tilde{z}$. Choose $k_1 > k_0$ such that for all $k > k_1$: $\frac{U^{1+\sigma_k}(\tilde{z}) - U^1(\tilde{z})}{\sigma_k} \leq H(\tilde{z}) + c$ and $\frac{U^{1+\sigma_k}(r) - U^1(r)}{\sigma_k} \geq H(r) - c \geq H(\tilde{z}) + 2c$. For $k > k_1$ we now have: $U^{1+\sigma_k}(r) \geq \sigma_k(H(\tilde{z}) + 2c) + U^1(r) \geq \sigma_k(H(\tilde{z}) + c) + U^1(r) + c\sigma_k \geq U^{1+\sigma_k}(\tilde{z}) + c\sigma_k > U^{1+\sigma_k}(\tilde{z})$, a contradiction with what we have found above. This completes the proof of *(d)*.

As to *(b)*, we still have to prove this statement for the case that U^1 has a unique global minimum location. Suppose x is a limit point of $\{g(\alpha)\}_{\alpha\downarrow 1}$, that is, there exist $\alpha_1, \alpha_2, \ldots$ with $\lim_{k\to\infty} \alpha_k = 1$ and $\lim_{k\to\infty} g(\alpha_k) = x$. Just as above, using Lemma 2, we derive $U^1(g(1)) = U^1(x)$, so $x = g(1)$ by uniqueness. Since this holds for any limit point of $g(\alpha)$ $(\alpha \downarrow 1)$ and since all $g(\alpha)$ are in the (compact) convex hull of $\{w^i : i \in N\}$, we conclude $g(1) = \lim_{\alpha\downarrow 1} g(\alpha) = v^1 \in \varphi^1(P)$. This completes the proof of *(b)*, and *(c)* is immediate. □

Proof of Theorem 4. Since, for $\alpha \geq 2$, U^α is differentiable everywhere, it is sufficient to set the gradient of U^α equal to 0 in order to find the unique global minimum location (recall that U^α is strictly convex for $\alpha > 1$, cf. Theorem 1 *(c)*). The gradient of U^α is the vector $\Sigma_{i=1}^n \varrho^i \|x - w^i\|_2^{\alpha-2}(x - w^i)$. *Taking* $\alpha = 2$ and setting the gradient equal to 0, we obtain $\Sigma_{i=1}^n \varrho^i(x - w^i) = 0$, hence $v^2 = \frac{\Sigma_{i=1}^n \varrho^i w^i}{\Sigma_{i=1}^n \varrho^i}$. So $\{v^2\} = \varphi^2(P)$. □

We conclude this section by proving some results that will imply Theorem 5.

Lemma 3. *Let $\|\cdot\|$ be any norm on $\mathbb{R}^m$, and let $M^\alpha : \mathbb{R}^m \to \mathbb{R}$ be defined by $M^\alpha(x) = \Sigma_{i=1}^n \varrho^i \|x - w^i\|^\alpha$. For $\alpha > 1$, let $t(\alpha)$ denote the (unique) global minimum location of M^α. Suppose there is a sequence $\alpha_1, \alpha_2, \ldots$ with $\lim_{k\to\infty} \alpha_k = \infty$ and $t := \lim_{k\to\infty} t(\alpha_k) \in \mathbb{R}^m_+$. Then $\min_{x\in\mathbb{R}^m} \max_{i\in N} \|x - w^i\| = \max_{i\in N} \|t - w^i\|$.*

Proof. Suppose not, i.e. $\max_{i\in N} \|t - w^i\| \geq c + 2\varepsilon$ for some $\varepsilon > 0$, where $c := \min_{x\in\mathbb{R}^m} \max_{i\in N} \|x - w^i\|$ (the existence of the min max is obvious, and left to the reader to prove). There is $k_0 \in \mathbb{N}$ such that $k \geq k_0 \Rightarrow \Sigma_{i\in N} \varrho^i \|t(\alpha_k) - w^i\|^{\alpha_k} \geq \varrho(c+\varepsilon)^{\alpha_k}$, where $\varrho := \min_{i\in N} \varrho^i$. Let $z \in \mathbb{R}^m$ such that $\max_{i\in N} \|z - w^i\| = \min_{x\in\mathbb{R}^m} \max_{i\in N} \|x - w^i\|$. Then $\Sigma_{i\in N} \varrho^i \|z - w^i\|^\alpha \leq \Sigma_{i\in N} \varrho^i c^\alpha$ for all α. Take $k_1 \geq k_0$ such that $k \geq k_1 \Rightarrow \Sigma_{i\in N} \varrho^i c^{\alpha_k} < \varrho(c+\varepsilon)^{\alpha_k}$. Then if $k \geq k_1$, we have: $M^{\alpha_k}(z) < M^{\alpha_k}(t(\alpha_k))$, a contradiction. □

Lemma 4. *Let M^α and $t(\alpha)$ be as in Lemma 3. Suppose furthermore that $\|\cdot\|$ is a strictly quasiconvex function, i.e. $\|y\| = \|z\| \Rightarrow \|\lambda y + (1-\lambda)z\| < \|y\|$ whenever $y, z \in \mathbb{R}^m$, $y \neq z$, and $0 < \lambda < 1$. Then $\lim_{\alpha\to\infty} t(\alpha)$ exists and is equal to the unique point t with $\max_{i\in N} \|t - w^i\| = \min_{x\in\mathbb{R}^m} \max_{i\in N} \|x - w^i\|$.*

Proof. In view of Lemma 3 and the fact that every $t(\alpha)$ is in the (compact) convex hull of the points w^i, it suffices to show that the point t is unique. Suppose $\max_{i\in N}\|s-w^i\| = \min_{x\in \mathbb{R}^m}\max_{i\in N}\|x-w^i\| =: c$ for some $s\in\mathbb{R}^m$. Let $\tilde{x} := \frac{1}{2}t+\frac{1}{2}s$ and suppose $\|\tilde{x}-w^j\|\geq\|\tilde{x}-w^i\|$ for all $i\in N$. Then $\|\tilde{x}-w^j\| = \left\|\frac{1}{2}(t-w^j)+\frac{1}{2}(s-w^j\right\| \leq \frac{1}{2}c+\frac{1}{2}c=c$. Since also $\|\tilde{x}-w^j\|\geq c$, we have $\|t-w^j\|=\|s-w^j\|=c$ and $\left\|\frac{1}{2}(t-w^j)+\frac{1}{2}(s-w^j)\right\| = \left\|\frac{1}{2}(t-w^j)\right\|+\left\|\frac{1}{2}(s-w^j)\right\|$. By strict quasiconvexity, we conclude $t-w^j=s-w^j$, hence $t=s$ as required. □

Theorem 5 now follows from Lemma 4 because $\|\cdot\|_2$ is strictly quasiconvex.

5. Conclusion

We have introduced a model for multiperson multicriteria decision making, and proposed a one-parameter family of solutions for such decision problems. The emphasis was on the intuitive contents of these solutions, inspired by the analogy with statistical concepts on the one hand, and with the potential function in physics on the other hand. A main concern for future research will be the development of an axiomatic basis for these solutions.

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